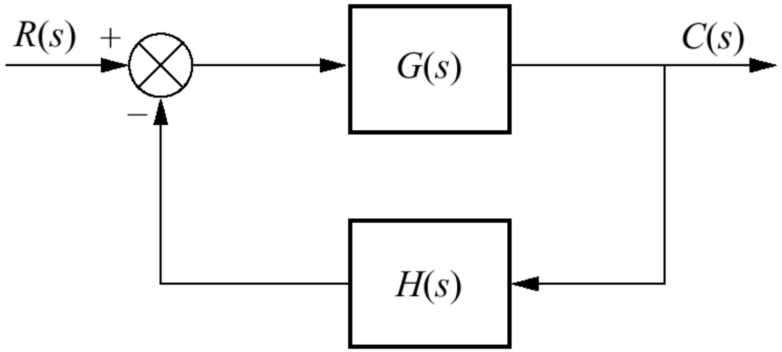
Frequency Response Method

Knowledge Before Studying Nyquist Criterion



$$T(s) = \frac{G(s)}{1+G(s)H(s)}$$

unstable if there is any pole on RHP (right half plane)

$$G(s) = \frac{N_G(s)}{D_G(s)} \qquad H(s) = \frac{N_H(s)}{D_H(s)}$$

Open-loop system:

$$G(s)H(s) = \frac{N_G(s)}{D_G(s)} \frac{N_H(s)}{D_H(s)}$$

Characteristic equation:

$$1 + G(s)H(s) = 1 + \frac{N_G N_H}{D_G D_H} = \frac{D_G D_H + N_G N_H}{D_G D_H}$$

poles of G(s)H(s) and 1+G(s)H(s) are the same

Closed-loop system:

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{N_G(s) D_H(s)}{D_G(s) D_H(s) + N_G(s) N_H(s)}$$

zero of 1+G(s)H(s) is pole of T(s)

$$G(s)H(s) = \frac{(s-1)(s-2)(s-3)(s-4)}{(s-5)(s-6)(s-7)(s-8)}$$

$G(s)H(s)$	$1+G(s)H(s)$	$\dfrac{G(s)}{1+G(s)H(s)}$
Zero – 1,2,3,4	**Zero – a,b,c,d**	**Zero – ?,?,?,?**
Poles – 5,6,7,8	**Poles – 5,6,7,8**	**Poles – a,b,c,d**

To know stability, we have to know a,b,c,d

Stability from Nyquist plot

From a Nyquist plot, we can tell a number of closed-loop poles on the right half plane.

- If there is any closed-loop pole on the right half plane, the system goes unstable.
- If there is no closed-loop pole on the right half plane, the system is stable.

Nyquist Criterion

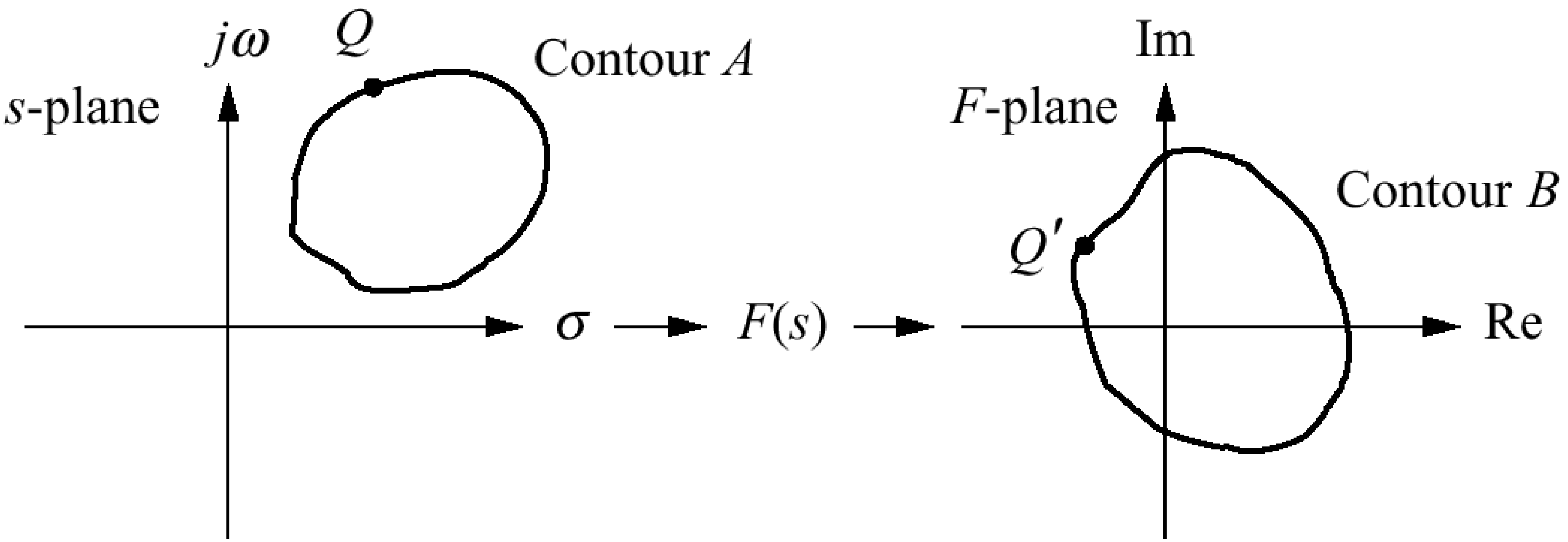
Nyquist plot is a plot used to verify stability of the system.

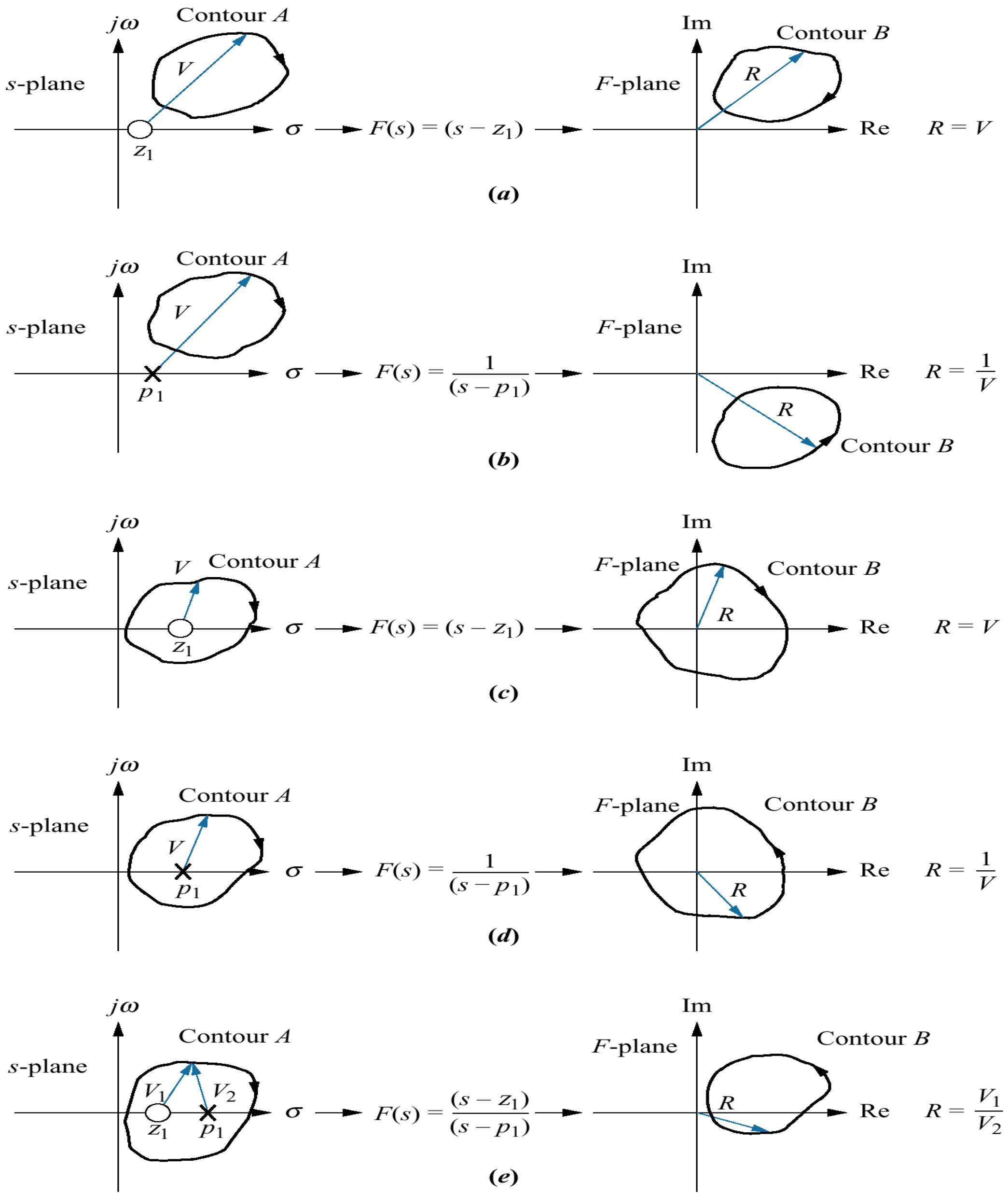
mapping contour

function $F(s) = \dfrac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)}$

mapping all points (contour) from one plane to another by function F(s).

$$F(s) = \frac{(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)}$$

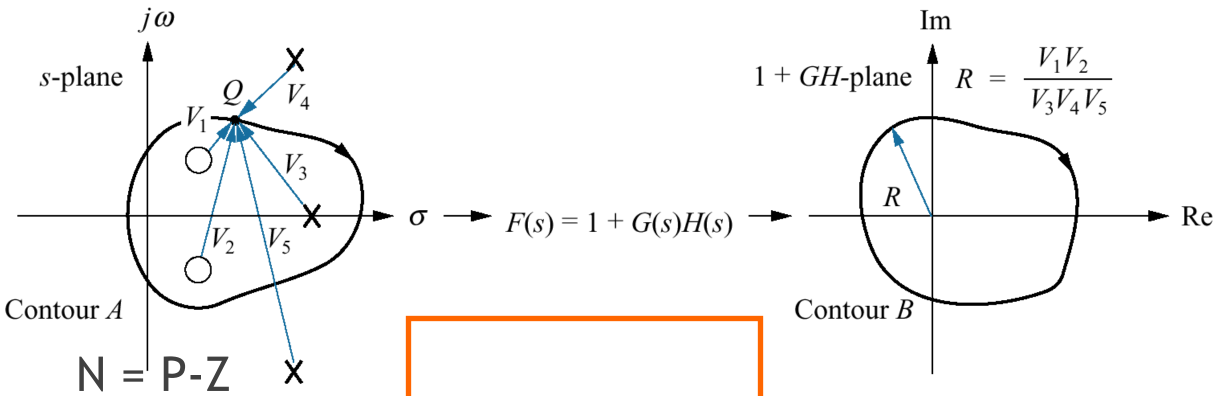




- Pole/zero inside the contour has 360 deg. angular change.
- Pole/zero outside contour has 0 deg. angular change.
- Move clockwise around contour, zero inside yields rotation in clockwise, pole inside yields rotation in counterclockwise

Characteristic equation

$$F(s) = 1 + G(s)H(s)$$



N = P-Z

N = # of counterclockwise direction about the origin

P = # of poles of characteristic equation inside contour

= # of poles of open-loop system

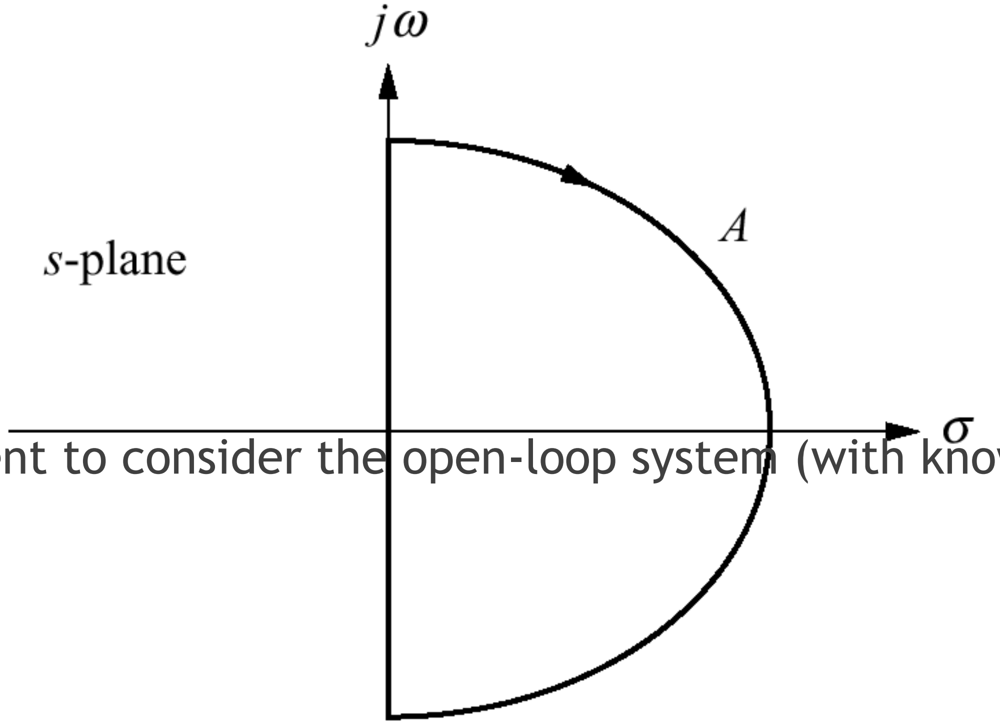
z = # of zeros of characteristic equation inside contour

= # of poles of closed-loop system

Z = P-N

Characteristic equation

- Increase size of the contour to cover the right half plane



- More convenient to consider the open-loop system (with known pole/zero)

Nyquist diagram of $G(s)H(s)$

'Open-loop system'

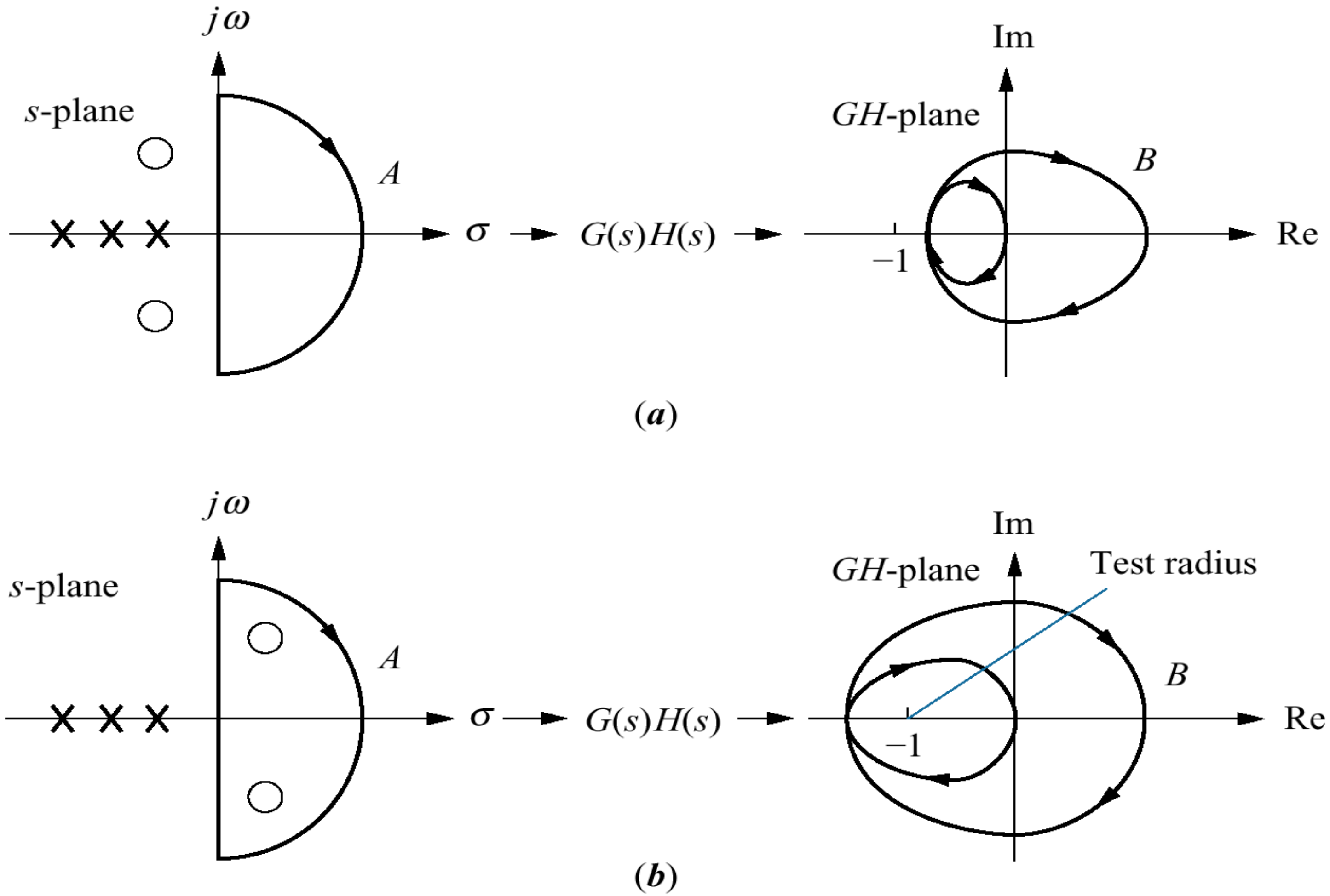
Mapping from characteristic equ. to open-loop system by shifting to the left one step

$$Z = P-N$$

Z = # of closed-loop poles inside the right half plane

P = # of open-loop poles inside the right half plane

N = # of counterclockwise revolutions around -1

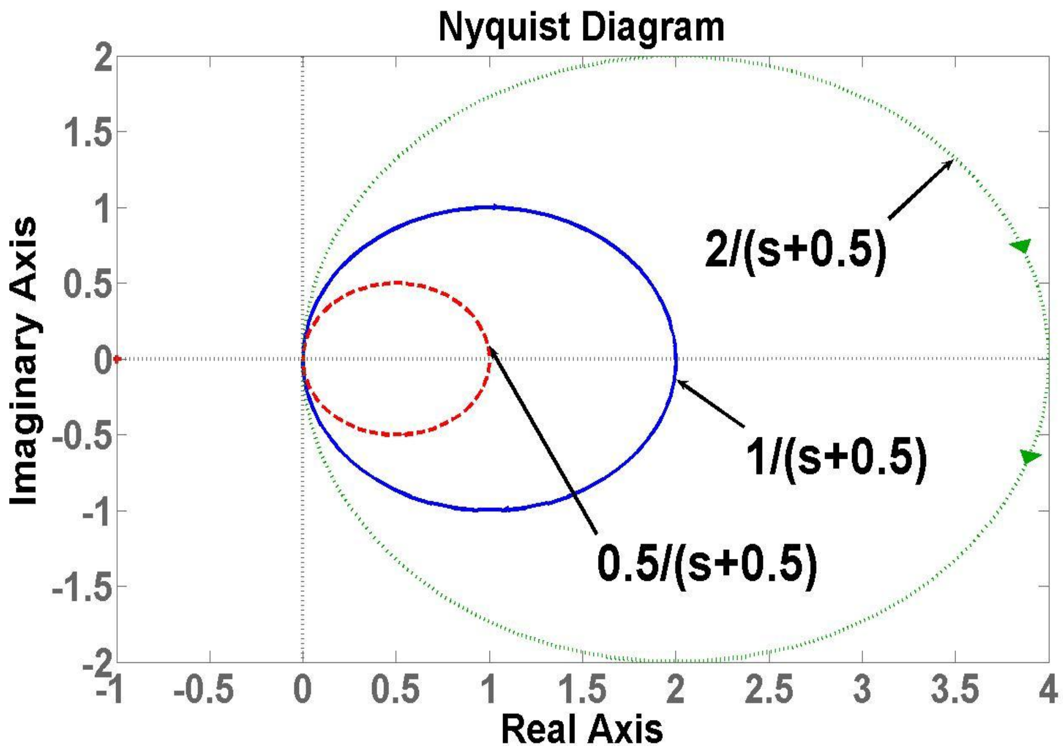


○ = zeros of $1 + G(s)H(s)$
= poles of closed-loop system
Location not known

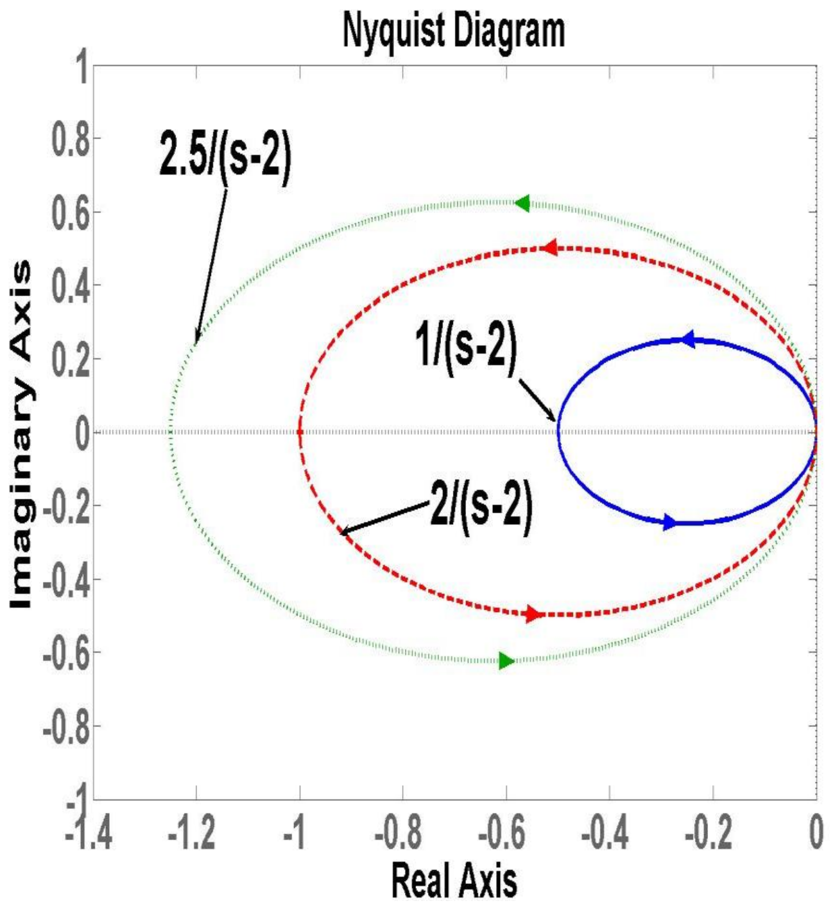
× = poles of $1 + G(s)H(s)$
= poles of $G(s)H(s)$
Location is known

Properties of Nyquist plot

If there is a gain, K, in front of open-loop transfer function, the Nyquist plot will expand by a factor of K.



Nyquist plot example



- Open loop system has pole at 2

$$G(s) = \frac{1}{s-2}$$

- Closed-loop system has pole at 1

$$\frac{G(s)}{1+G(S)} = \frac{1}{(s-1)}$$

- If we multiply the open-loop with a gain, K, then we can move the closed-loop pole's position to the left-half plane

Nyquist plot example (cont.)

- New look of open-loop system:

$$G(s) = \frac{K}{s-2}$$

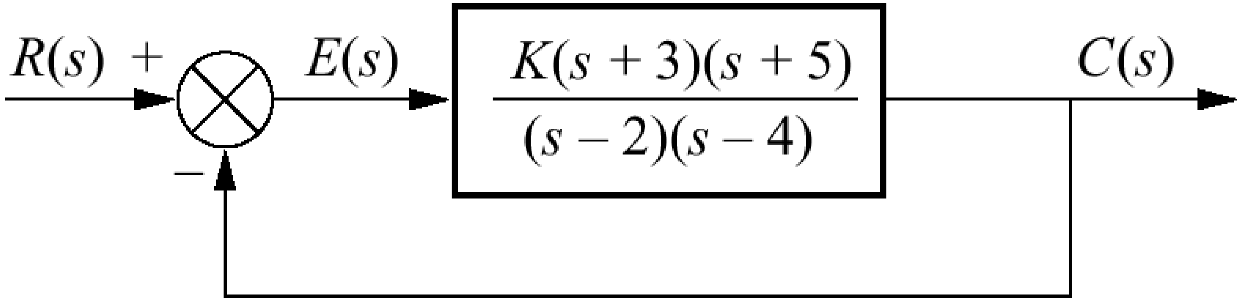
- Corresponding closed-loop system:

$$\frac{G(s)}{1+G(s)} = \frac{K}{s+(K-2)}$$

- Evaluate value of K for stability

$$\boxed{K \geq 2}$$

Adjusting an open-loop gain to guarantee stability



(a)

Step I: sketch a Nyquist Diagram
Step II: find a range of K that makes the system stable!

How to make a Nyquist plot?

Easy way by Matlab

- Nyquist: ‘nyquist’
- Bode: ‘bode’

Step I: make a Nyquist plot

- Starts from an open-loop transfer function (set K=1)
- Set $s = j\omega$ and find frequency response
  - At dc, $\omega = 0 \rightarrow s = 0$
  - **Find ω at which the imaginary part equals zero**

$$G(s)H(s)=\frac{(s+3)(s+5)}{(s-2)(s-4)}=\frac{s^2+8s+15}{s^2-6s+8}$$

$$G(j\omega)H(j\omega)=\frac{-\omega^2+8j\omega+15}{-\omega^2-6j\omega+8}=\frac{(15-\omega^2)+8j\omega}{(8-\omega^2)-6j\omega}$$

$$=\frac{(15-\omega^2)+8j\omega}{(8-\omega^2)-6j\omega}\times\frac{(8-\omega^2)+6j\omega}{(8-\omega^2)+6j\omega}$$

$$=\frac{(15-\omega^2)(8-\omega^2)-48\omega^2+j(154\omega-14\omega^3)}{(8-\omega^2)^2+6^2\omega^2}$$

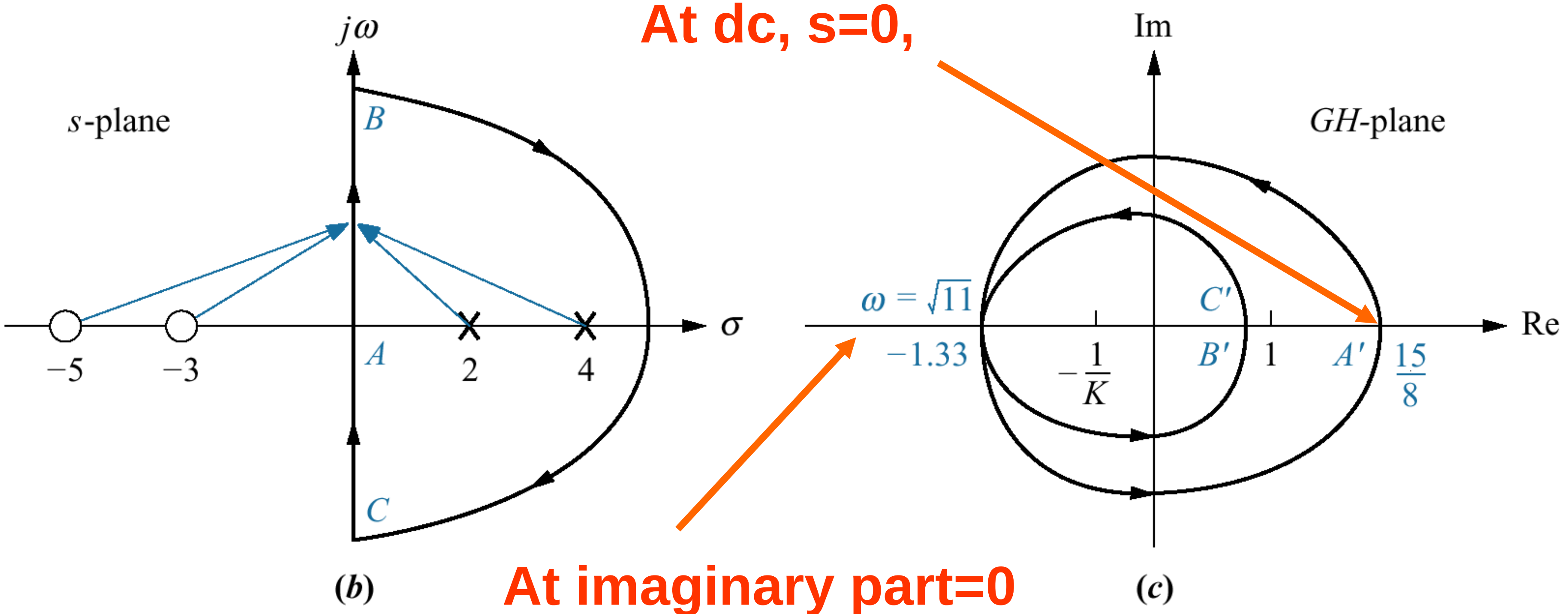
$\omega=0,\sqrt{11}$

Need the imaginary term = 0,

Substitute $\omega=\sqrt{11}$ **back in to the transfer function**
And get $G(s)=-1.33$

$$\frac{(15-11)(8-11)-48(11)}{(8-11)^2+6^2(11)}=\frac{-540}{412}=-1.31$$

At dc, s=0,



At imaginary part=0

Step II: satisfying stability condition

- **P = 2, N has to be 2 to guarantee stability**
- **Marginally stable if the plot intersects -1**
- **For stability, 1.33K has to be greater than 1**

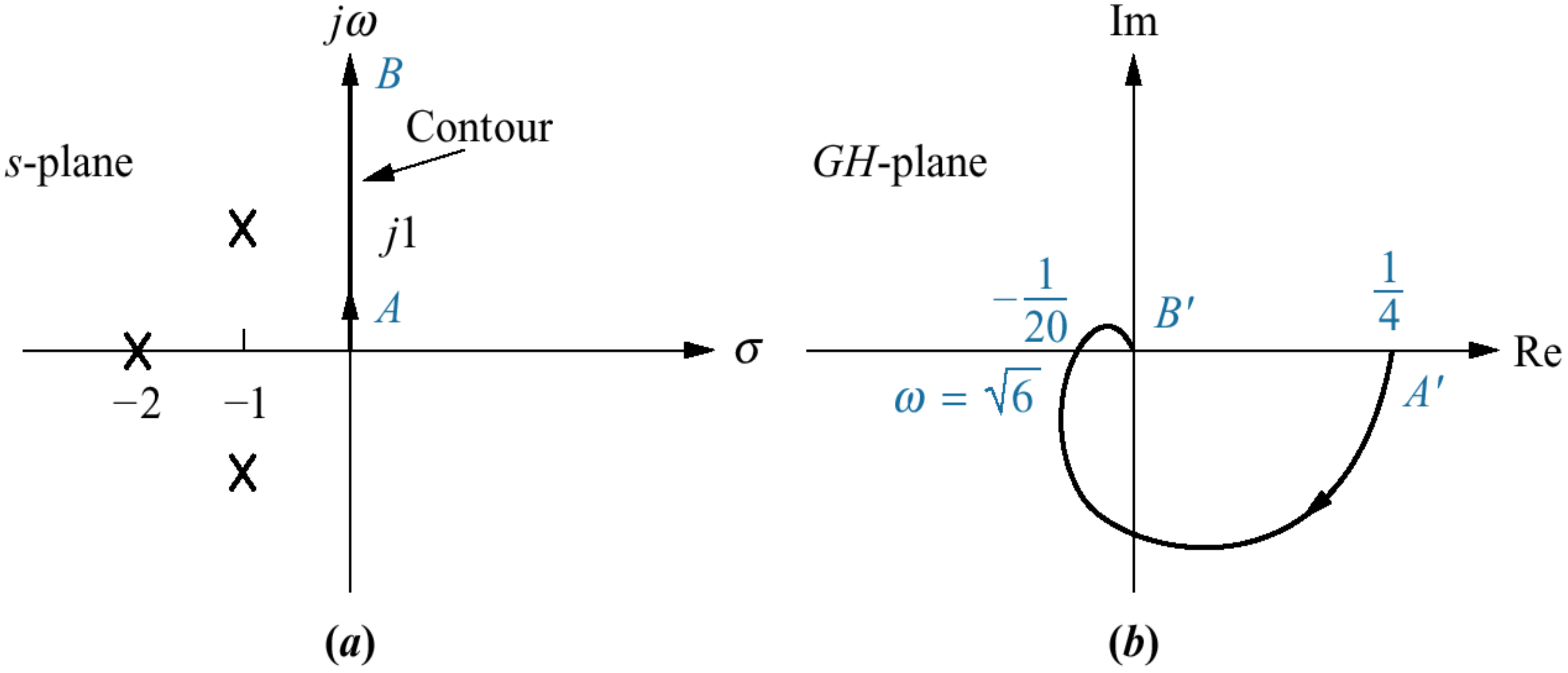
$$K > 1/1.33$$

$$\text{or} \quad K > 0.75$$

Example

Evaluate a range of K that makes the system stable

$$G(s) = \frac{K}{(s^2 + 2s + 2)(s + 2)}$$



Step I: find frequency at which imaginary part = 0

Set $S = j\omega$

$$G(j\omega) = \frac{K}{((j\omega)^2 + 2j\omega + 2)(j\omega + 2)}$$

$$= \frac{4(1-\omega^2) - j\omega(6-\omega^2)}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2}$$

At $\omega = 0, \sqrt{6}$ **the imaginary part = 0**

Plug $\omega = \sqrt{6}$ **back in the transfer function and get G = -0.05**

Step II: consider stability condition

- **P = 0, N has to be 0 to guarantee stability**
- **Marginally stable if the plot intersects -1**
- **For stability, 0.05K has to be less than 1**

$$K < 1/0.05$$

or $$K < 20$$

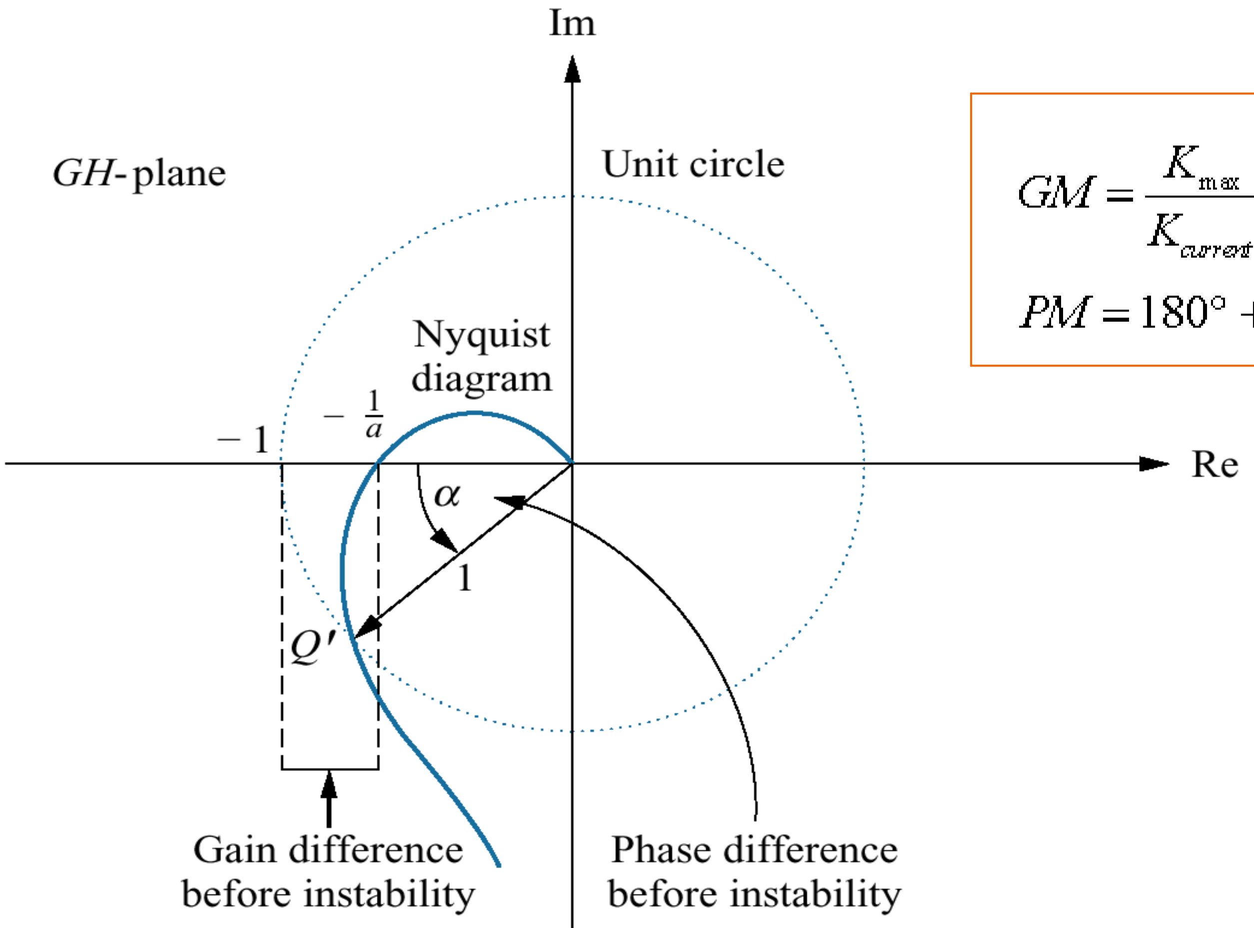
Gain Margin and Phase Margin

Gain margin is the change in open-loop gain (in dB), required at 180 of phase shift to make the closed-loop system unstable.

Phase margin is the change in open-loop phase shift, required at unity gain to make the closed-loop system unstable.

GM/PM tells how much system can tolerate before going unstable!!!

GM and PM via Nyquist plot

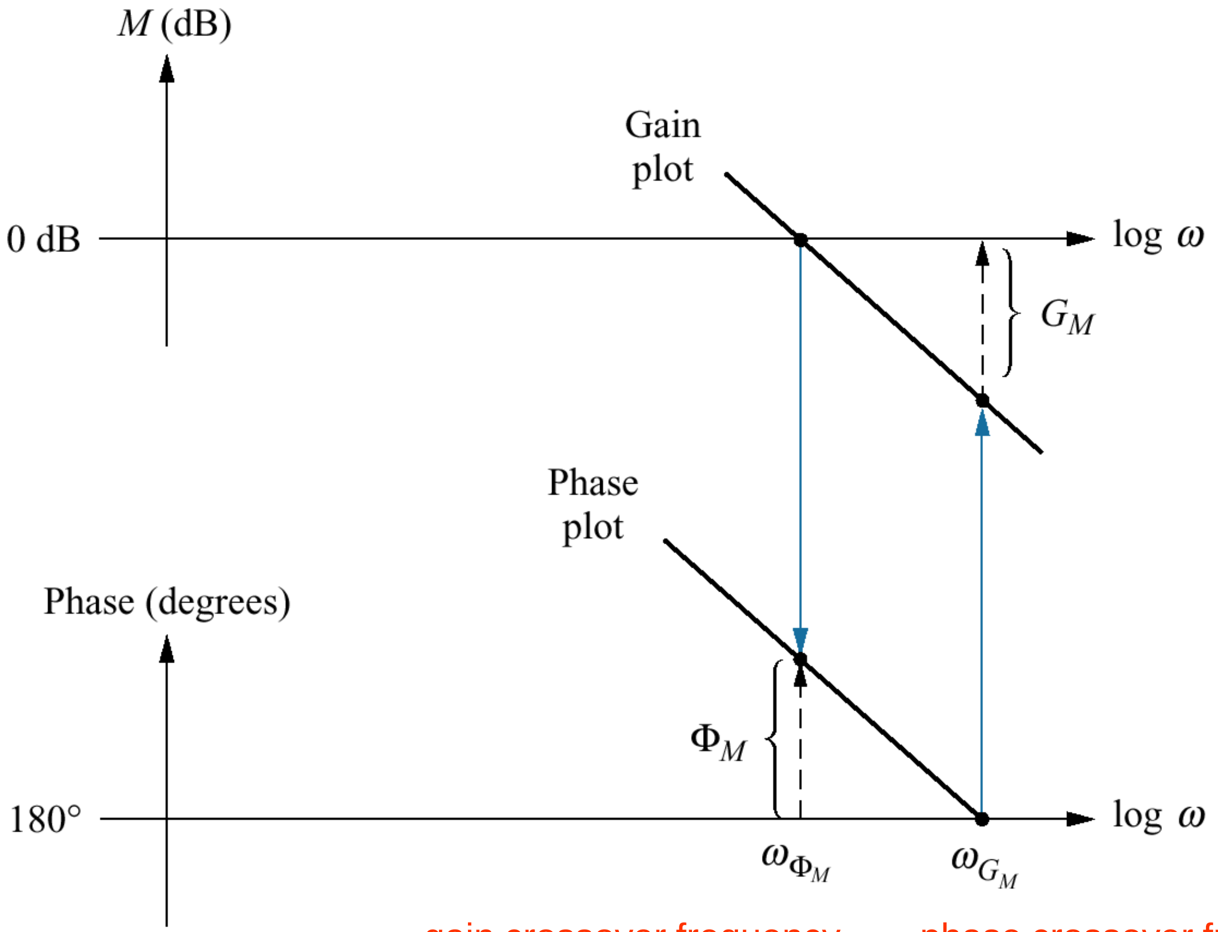


Gain margin = $G_M = 20 \log a$ Phase margin = $\Phi_M = \alpha$

$$GM = \frac{K_{max}}{K_{current}} = \frac{1}{|G(j\omega_\phi)H(j\omega_\phi)|}$$

$$PM = 180^\circ + \angle G(j\omega_x)H(j\omega_x)$$

GM and PM via Bode Plot



gain crossover frequency   phase crossover frequency

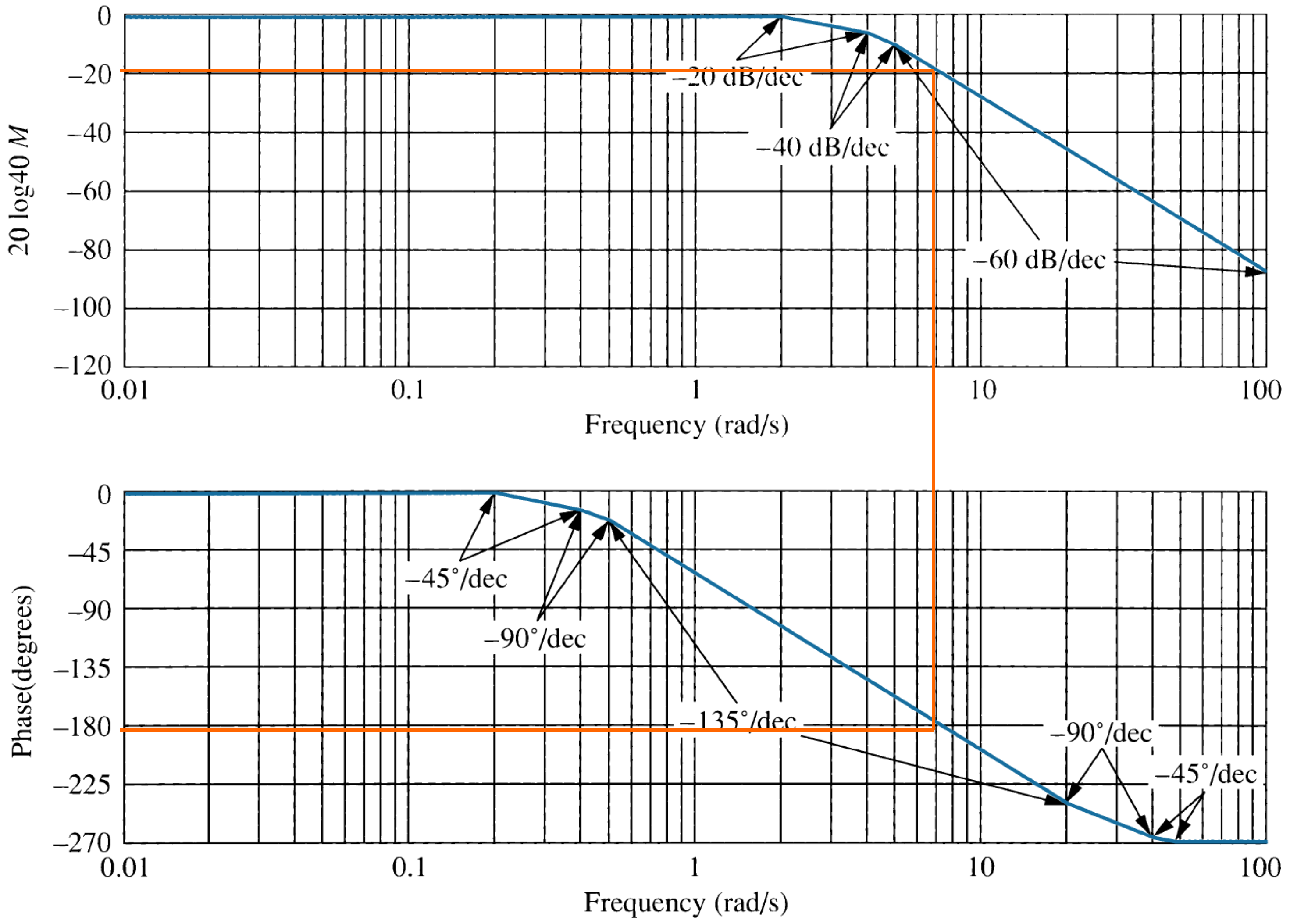
- The frequency at which the phase equals 180 degrees is called the phase crossover frequency ω_{G_M}
- The frequency at which the magnitude equals 1 is called the gain crossover frequency

Example

Find Bode Plot and evaluate a value of K that makes the system stable
The system has a unity feedback with an open-loop transfer function

$$G(s) = \frac{K}{(s+2)(s+4)(s+5)}$$

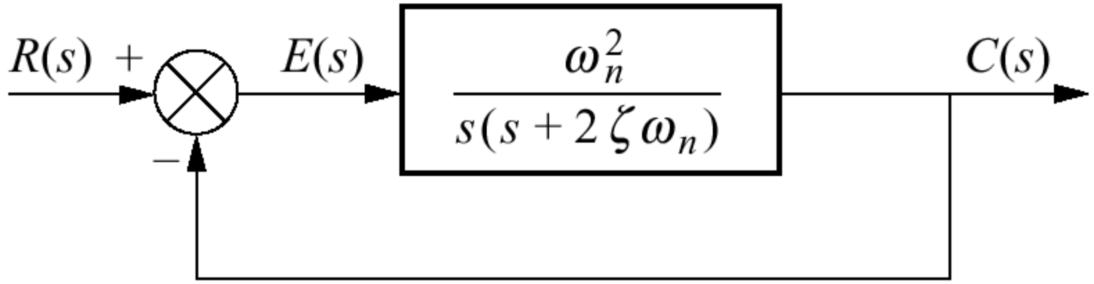
First, let's find Bode Plot of G(s) by assuming that K=40 (the value at which magnitude plot starts from 0 dB)



At phase = -180, ω = 7 rad/sec, magnitude = -20 dB

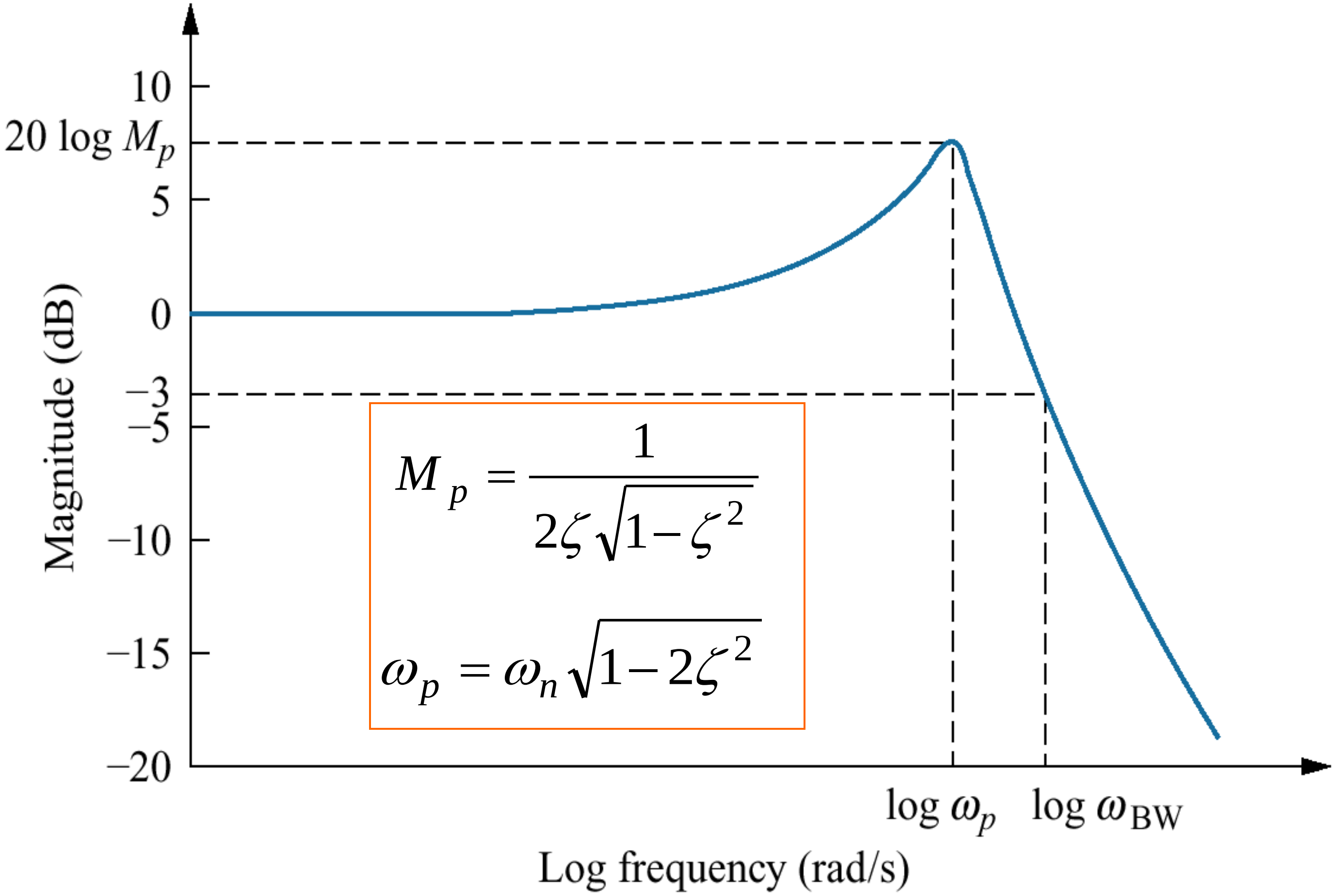
- GM>0, system is stable!!!
- Can increase gain up 20 dB without causing instability (20dB = 10)
- Start from K = 40
- with K < 400, system is stable

Closed-loop transient and closed-loop frequency responses
'2nd system'



$$\frac{C(s)}{R(s)} = T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Damping ratio and closed-loop frequency response



Magnitude Plot of closed-loop system

Response speed and closed-loop frequency response

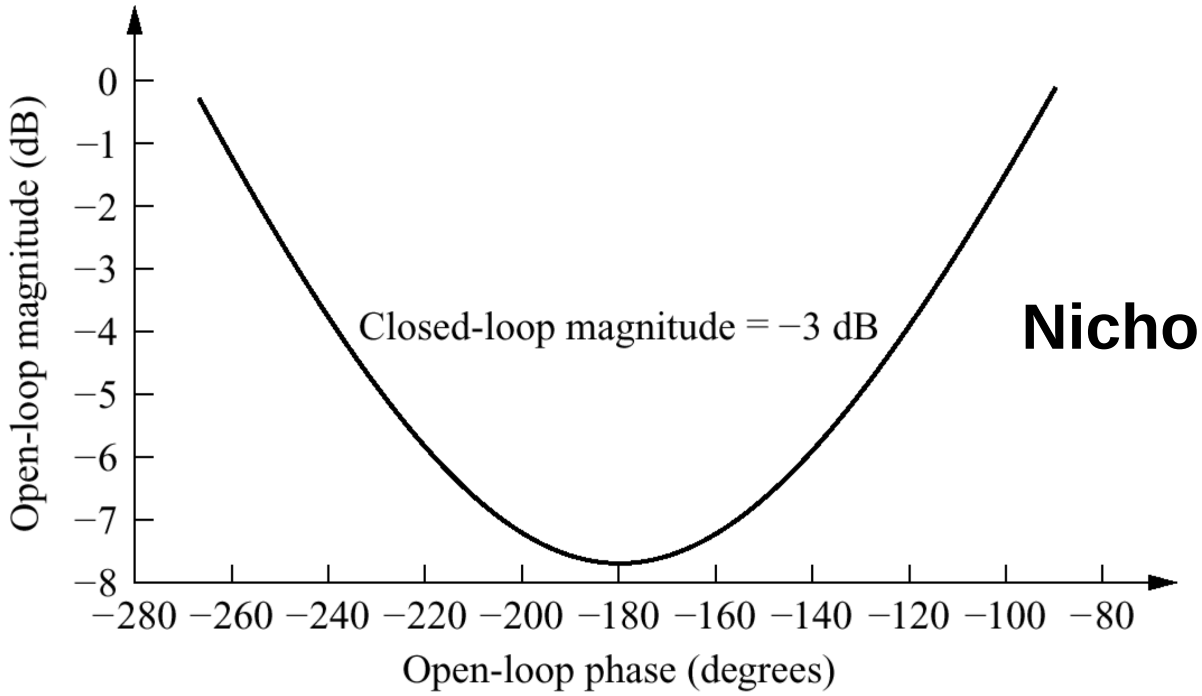
$$\omega_{BW} = \omega_n \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

$$\omega_{BW} = \frac{4}{T_s \zeta} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

$$\omega_{BW} = \frac{\pi}{T_p \sqrt{1-\zeta^2}} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

ω_{BW} **= frequency at which magnitude is 3dB down from value at dc (0 rad/sec), or** $M = \frac{1}{\sqrt{2}}$ **.**

Find ω_{BW} from Open-loop l



Nichols Charts

From open-loop frequency response, we can find ω_{BW} at the open-loop frequency that the magnitude lies between -6dB to -7.5dB (phase between -135 to -225)

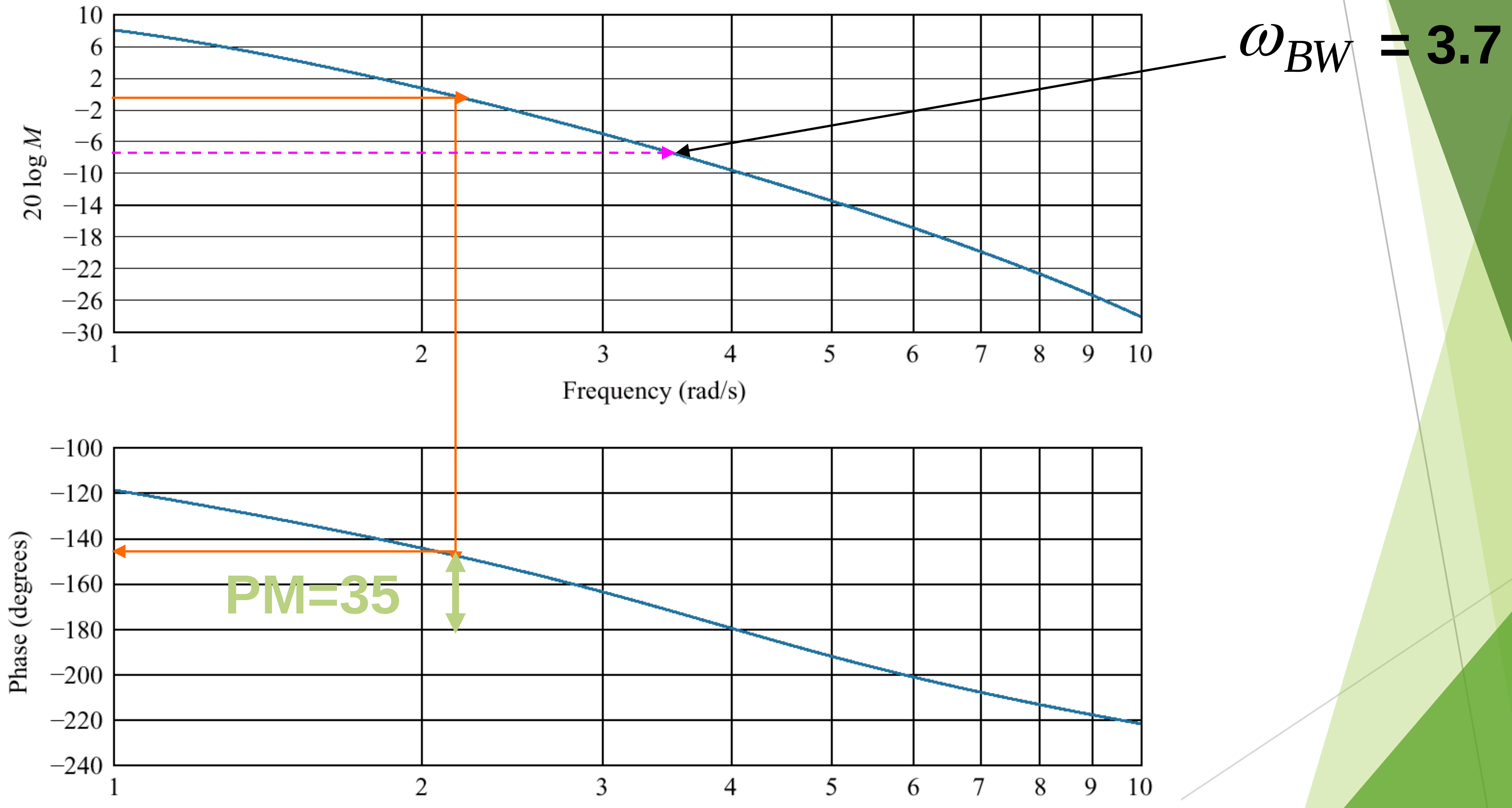
Relationship between damping ratio and phase margin of open-loop frequency response

Phase margin of open-loop frequency response
Can be written in terms of damping ratio as following

$$\phi_M = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}$$

Example

Open-loop system with a unity feedback has a bode plot below, approximate settling time and peak time



$$\phi_M = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}$$

Solve for PM = 35 $\zeta = 0.32$

$$T_s = \frac{4}{\omega_{BW}\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

$$= 5.5$$

$$T_p = \frac{\pi}{\omega_{BW}\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

$$= 1.43$$